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A NEW APPROACH TO THE ASYMPTOTIC INTEGRATION OF THE EQUATIONS OF SHALLOW CONVEX SHELL THEORY IN THE POST-CRITICAL STAGE*

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A method is proposed for the asymptotic integration of the non-linear equations of shallow elastic shell theory on the basis of a new definition of the small parameter that is selected to be proportional to the ratio between the shell thickness and the amplitude of its deflection. This parameter is actually small if the shell is in the post-critical stage, i.e., its deflections are large. An asymptotic expansion of the solution of the shell equilibrium equations in the parameter mentioned is carried out. It is established that the first two approximations result in the geometric theory of shell stability formulated by Pogorelov /1/. By comparing the asymptotic and numerical solutions /2/ found for a spherical shell under axisymmetric deformation, satisfactory accuracy of the proposed method is obtained for fairly large deflection. The well-known Koiter approach is used in the small-deflection domain. The two asymptotic expansions, one of which is suitable for small deflections and the other for large, are merged using the Padé approximation.

Despite the efficiency of the well-known asymptotic method (/3-5/, etc.) in non-linear shell theory, the singularities of the non-linear equations describing the behaviour of the shell for deflections substantially exceeding its thickness are not used therein. The significant post-critical shell deformations are described well in a number of cases by the Pogorelov /1/ geometric theory which is, however, phenomenological in nature. The investigations in /3-7/ are devoted to proving the geometrical method. The paper by Lesnichaya /7/ should be noted, in which the ratio between the shell thickness and the characteristic dimension of the domain of the post-critical dents is utilized as the small parameter in a study of the axisymmetric deformation of a closed sphere under uniform external pressure. Relationships of the geometrical theory are obtained as the fundamental approximation. However, the connection

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between the parameters of the post-critical equilibrium mode and the magnitude of the load has not been established.

The distinguishing feature of the approach proposed below is the conversion of the system of resolving equations of shallow shell theory by the introduction of new variables that are disclosed in the examination of the bending of the original middle surface with violation of the regularity along lines. These lines, as well as those orthogonal to them, are taken as coordinate lines. Consequently, a new small parameter is discovered that directly characterises the non-linearity of the system.

1. Within the framework of shallow shell theory the post-critical axisymmetric deformation of a closed sphere is examined under a uniform external pressure q. The initial resolving equations have the form

$$\frac{D}{h}\frac{d}{dr}(\nabla^2 W) = \frac{1}{R}\frac{d\Phi}{dr} + \frac{1}{r}\frac{dW}{dr}\frac{d\Phi}{dr} + \frac{rq}{2h} \tag{1.1}$$
$$\frac{d}{dr}(\nabla^2 \Phi) = -E\left[\frac{1}{R}\frac{dW}{dr} + \frac{1}{2r}\left(\frac{dW}{dr}\right)^2\right], \quad D = \frac{Eh^3}{12(1-\nu^2)}$$

where Φ is the stress function, R is the sphere radius, and E and ν are the elastic modulus and Poisson's ratio of the material. System (1.1) allows of two obvious solutions. The first

$$W = \text{const}, \quad d\Phi/dr = -qrR/(2h)$$

corresponds to the initial membrane state of the shell. The second describes isometric transformation of the sphere obtained by the specular reflection of a segment with respect to the plane of its base /1/, and has the form

$$W = W^\circ(1 - r^2/(W^\circ R)) \tag{1.2}$$

Introducing the change of variables

$$z = r^2/(W^\circ R), \quad w = W/W^\circ$$

corresponding to relationship (1.2), we arrive at the equations

$$\varepsilon^2 \frac{d^2}{dz^2}\left(z\frac{dw}{dz}\right) = \frac{d\varphi}{dz}\left(1 + 2\frac{dw}{dz}\right) + q^\circ \tag{1.3}$$
$$\varepsilon^2 \frac{d^2}{dz^2}\left(z\frac{d\varphi}{dz}\right) = -\frac{dw}{dz}\left(1 + \frac{dw}{dz}\right)$$
$$\varepsilon^2 = \frac{2}{w^\circ\sqrt{3(1-\nu^2)}}, \quad w^\circ = \frac{W^\circ}{h}, \quad \Phi = \varphi\frac{EhW^\circ}{\sqrt{12(1-\nu^2)}}$$
$$q^\circ = \frac{q}{q_*}, \quad q_* = \frac{2E}{\sqrt{3(1-\nu^2)}}\left(\frac{h}{R}\right)^2$$

A feature of the system obtained is the presence of the parameter ε that decreases as the deflection amplitude W° increases and becomes small for substantially post-critical configurations. The limit system of equations (for $\varepsilon = 0$) has two solutions. The first corresponds to the original membrane state of the shell and the second to an isometric transformation of the middle surface that has the following very simple form in the new variable:

$$w = 1 - z \tag{1.4}$$

The composite solution for $z = 1$ undergoes a discontinuity which is compensated by interior boundary layer functions. Consequently, in conformity with /8/, taking account of the first approximations, the asymptotic expansion of the solution of the system (1.3) as $\varepsilon \to 0$ is sought in the form

$$w = \varepsilon^n w_{n-1}, \quad \varphi = \varepsilon^n \varphi_{n-1}, \quad q^\circ = q_0 + \varepsilon^n q_n \quad (n = 1, 2, 3, 4) \tag{1.5}$$
$$\varepsilon w_i = W_i(z) + \varepsilon v_i(t), \varepsilon\varphi_i = \Phi_i(z) + \varepsilon u_i(t), \; t = (1-z)/\varepsilon$$

where v_i and u_i are functions describing the internal edge effect, and W_i and Φ_i are functions corresponding to the fundamental state. It can be determined that

$$\frac{dW_k}{dz} = 0, \quad \frac{d\Phi_k}{dz} = -q_k, \quad z > 1 \tag{1.6}$$
$$\frac{dW_0}{dz} = -1, \quad \frac{dW_{k+1}}{dz} = 0, \quad \frac{d\Phi_k}{dz} = q_k, \quad z < 1 \quad (k = 0, 1, 2, 3)$$

It is convenient to represent the components of the solutions W_k and Φ_k in the form of functions which depend on the variable t. For instance, for $z < 1$ we have $W_0 = 1 - z = \varepsilon t$.

Then the w_i and φ_i in (1.5) can be considered to be functions of the variable t which are continuous together with their derivatives and, in conformity with relationships (1.6), should satisfy the following boundary conditions

$$w_k' = 0, \quad \varphi_k' = q_k, \quad t \to -\infty; \quad w_0' = 1, \quad w_{k+1}' = 0, \quad \varphi_k' = -q_k, \quad t \to +\infty \tag{1.7}$$

Taking account of the expansion (1.5) after the asymptotic analysis of (1.3), we obtain the following equations:

in the fundamental approximation

$$w_0''' - \varphi_0' (1 - 2w_0') + q_0 = 0 \tag{1.8}$$

$$\varphi_0''' + w_0' (1 - w_0') = 0 \tag{1.9}$$

in the second approximation

$$w_1''' - (tw_0')''' + 2w_1'\varphi_0' - \varphi_1' (1 - 2w_0') + q_1 = 0 \tag{1.10}$$

$$\varphi_1''' - 2\varphi_0'' - t\varphi_0'' + w_1' (1 - 2w_0') = 0 \tag{1.11}$$

in the third approximation

$$w_2''' - (tw_1')'' + 2w_1'\varphi_1' - \varphi_2' (1 - 2w_0') + 2\varphi_0'w_2' + q_2 = 0 \tag{1.12}$$

$$\varphi_2''' - 2\varphi_1'' - t\varphi_1''' + w_2' (1 - 2w_0') - w_1'^2 = 0 \tag{1.13}$$

In combination with the boundary conditions (1.7) the equations presented can obviously be used to determine the functions w_i and φ_i for any given values of q_i. The equations are linear in the second and subsequent approximations. However, the coefficients themselves of the load expansion in a series in the parameter ε remain undetermined. The reason for this indeterminacy becomes clear if we return to the appropriate variational formulation of the problem.

Let us examine the functional of the total shell potential energy, which, after asymptotic analysis in conformity with expansions (1.5), acquires the form

$$U = D_1 [J_0\varepsilon + J_1\varepsilon^2 + J_2\varepsilon^3 - q^\circ (1 + 2\varepsilon^2 \int v_0 dt + 2\varepsilon^3 \int w_1 dt) + O(\varepsilon^4)] \tag{1.14}$$

$$J_0 = \int (\varphi_0''^2 + w_0''^2)\, dt, \quad J_1 = 2 \int [\varphi_0'' (\varphi_1'' - t\varphi_0'') + w_0'' (w_1'' - tw_0'')]\, dt$$

$$J_2 = \int (\varphi_1''^2 + 2\varphi_0''\varphi_2'' - 4t\varphi_0''\varphi_1'' + t^2\varphi_0''^2 + w_1''^2 + 2w_0''w_2'' - 4tw_0''w_1'' + t^2 w_0''^2)\, dt, \quad D_1 = \frac{16\pi D}{\sqrt{3(1-\nu^2)}\, R\varepsilon^4} \tag{1.15}$$

Here and everywhere later the integration is between the limits $-\infty$ and $+\infty$.

Taking account of the representation of q° in the form of the series (1.5), we obtain

$$U = D_1 [-q_0 + \varepsilon (I_0 - q_1) + \varepsilon^2 (I_1 - q_2) + \varepsilon^3 (I_2 - q_3) + O(\varepsilon^4)] \tag{1.16}$$

$$I_0 = J_0, \quad I_1 = J_1 - 2q_0 \int v_0 dt, \quad I_2 = J_2 - 2q_1 \int v_0 dt - 2q_0 \int w_1 dt$$

It can be shown that the variation of the total potential energy functional (1.14) in the functions w_i and φ_i, taking the constraints (1.9), (1.11) and (1.13) and the boundary conditions (1.7) into account in each approximation as the Euler equations, yields the appropriate equilibrium Eqs.(1.8), (1.10) and (1.12). For example, considering the problem of the minimum of the functional I_2 in the presence of the constraints (1.9), (1.11) and (1.13) during its variation in the functions w_1 and φ_1, we arrive at relationship (1.10), while varying the functional I_2 in the functions w_0 and φ_0 we obtain (1.12). However, the parameter ε remains the same here, but should also be considered and variational, since it is related to the amplitude of the post-critical configuration deflection. Varying (1.14) in ε we obtain the relationships

$$q_0 = 0, \quad q_1 = \frac{3}{4} J_0, \quad q_2 = \frac{1}{2} J_1, \quad q_3 = \frac{1}{4} J_2 - q_1 \int v_0\, dt$$

where J_i should be understood to be the minimum value of these functionals. Relationships (1.7)-(1.13) possess symmetry which enable us to conclude that the functions φ_0', φ_2', w_1' are even while φ_1', w_0', w_2' are odd. It hence follows that $J_1 = 0$ since the appropriate integrand is odd. Then $q_2 = 0$. It can be established that $q_4 = 0$.

The components containing w_2'' and φ_2'' in (1.15) can be integrated. Then taking account of (1.7) we obtain

$$J_2 = \int [\varphi_1''^2 + t^2\varphi_0''^2 - 4tw_0''w_1'' + t^2 w_0''^2 + 2\varphi_0' (t\varphi_1''' - w_1'^2)]\, dt \tag{1.17}$$

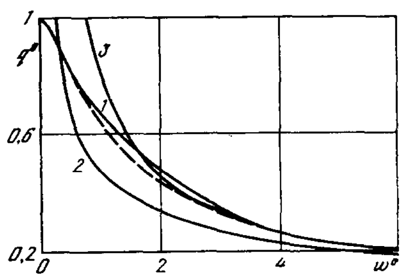
Hence it follows that to determine q_1 it is necessary to integrate (1.8) and (1.9) in the fundamental approximation. The coefficient q_3 will also be determined by the functions w_1 and φ_1 of the second approximation.

For an appropriate change of variables the functional J_0 reduces to the Pogorelov functional, whose minimum is $J_* = 2J_0 \simeq 1.12$. Solving the problem of the minimum of the functional J_2 by using the Ritz method, we obtain approximately $J_2 = -0.4$. Finally, we arrive at the relationship (for $\nu = 0.3$)

$$q^\circ = 0.42\varepsilon + 0.26\varepsilon^3 + O(\varepsilon^5) \tag{1.18}$$

Apart from the factor $(1 - \nu^2)^{1/4}$ the first component yields the well-known result in /1/. Therefore, it is established that the relationships of the geometric theory are asymptotically exact for $\varepsilon \to 0$ taking the first two approximations into account.

The result obtained is represented in the form of graphs in the figure. Curve *1* corresponds to the exact solution obtained numerically /2/. Curve *2* is obtained taking the fundamental approximation into account, which corresponds to the geometric theory. Formula (1.18) is represented by curve *3* in the graph. It follows from a comparison of curves *1* and *3* that there is good agreement between the data for $h/W^\circ \lesssim 1$. As $W^\circ \to 0$ the asymptotic approach under consideration yields a qualitatively false result. However, in this domain we apply the fairly well-developed Koiter approach, by means of which we obtain the following asymptotic formula by using the perturbation method for small deflections



$$q^\circ = 1 + aw^\circ + O(w^{\circ 2}) \tag{1.19}$$

where $a = 0$ for the axisymmetric deformation of a shallow sphere under external pressure. Since a quantity reciprocal to ε is considered as the small parameter here, expression (1.19) yields the first terms of the series in the expansion of the function $q^\circ(\varepsilon)$ in powers of $1/\varepsilon$

We merge the asymptotic expansions (1.18) and (1.19) by using Padé two-point approximations /9/. For this $q^\circ(\varepsilon)$ is sought in the form of a rational-fraction function whose coefficients are determined from the condition for the expansions of this function to agree, as $\varepsilon \to 0$ and $\varepsilon \to \infty$, with the expansions (1.18) and (1.19), respectively. We finally obtain the dependence

$$q^\circ(\varepsilon) = A(\varepsilon)/(1 + A(\varepsilon)) \tag{1.20}$$
$$A(\varepsilon) = 0.42\varepsilon + 0.176\varepsilon^2 + 0.333\varepsilon^3 + 0.4\varepsilon^4$$

to which the dashed line in the figure corresponds. Comparison with the data in /2/ (curve *1*) indicates sufficient accuracy for the solution obtained.

2. The results presented can be extended to the case of strictly convex shallow shells with principal radii of curvature R_1 and R_2. We will limit ourselves to a more detailed examination of the fundamental approximation. The strain compatibility equation has the form

$$E^{-1}\nabla^4\Phi = W_{\alpha\beta}^2 - W_{\alpha\alpha}W_{\beta\beta} - W_{\alpha\alpha}/R_1 - W_{\beta\beta}/R_2 \tag{2.1}$$

The function

$$W = W^0 (1 - \alpha^2/(W^0R_1) - \beta^2/(W^\circ R_2)) \tag{2.2}$$

vanishes on the right-hand side of this equation and describes an isometric transformation of the specular reflection of the initial middle surface relative to a certain plane. We obtain a piecewise-smooth surface with regularity violated along lines in the plane under consideration. We take these lines as well as those orthogonal to them as the coordinate lines, which corresponds to the following change of variables

$$t_1 = \frac{\alpha^2}{W^0R_1} + \frac{\beta^2}{W^0R_2}, \quad t_2 = \frac{\alpha^2}{W^0R_1} - \frac{\beta^2}{W^0R_2} \tag{2.3}$$

Going over to dimensionless quantities w and φ we establish the presence of a small parameter in the initial relationships

$$\varepsilon^2 = \frac{c^2 (R_1 + R_2)^2}{\sqrt{12(1-\nu^2)}\, R_1 R_2 w^0}, \quad c = 1 + \frac{t_2}{t_1} \frac{R_2 - R_1}{R_2 + R_1}, \quad R_2 \geqslant R_1 \tag{2.4}$$

which coincides with that obtained for the spherical shell for $R_1 = R_2 = R$. After an asymptotic analysis of the total shell potential energy functional, we obtain

$$U = D_1 (J_0 \varepsilon - q^\circ) \tag{2.5}$$

$$D_1 = \frac{\pi (R_1 + R_2)^4 hD}{\sqrt{3(1-\nu^2)}\,(R_1 R_2)^{5/2} \varepsilon^4}, \quad q^\circ = \frac{q}{q_*}, \quad q_* = \frac{2Eh^2}{\sqrt{3(1-\nu^2)}\, R_1 R_2}$$

The function J_0 agrees in accuracy with that presented for a spherical shell. However, the need for the requirement $c \simeq 1$ for $t_1 = 1$ and $|t_2| \leqslant 1$ is here established in a natural manner, which imposes an additional constraint on the relationships obtained in the form

$$\varepsilon_1 = (R_2 - R_1)/(R_2 + R_1) \ll 1 \tag{2.6}$$

A numerical analysis shows that in practice it is sufficient to limit ourselves to the requirement $2R_2 \leqslant R_1$. Under these conditions we obtain the formula

$$q^\circ = 0.42\varepsilon + O(\varepsilon^3) + O(\varepsilon_1)$$

which corresponds to the result obtained in /1/. When constructing the solution in higher approximation we arrive at relationship (1.18). Using the procedure described to merge the solutions of large and small relative deflections, we obtain (1.20) in this case, in which

$$\varepsilon^2 = \frac{(R_1 + R_2)^2}{\sqrt{12(1-\nu^2)}\, R_1 R_2 w^0}$$

The simple relationships presented indicate the efficiency of the approach proposed.

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